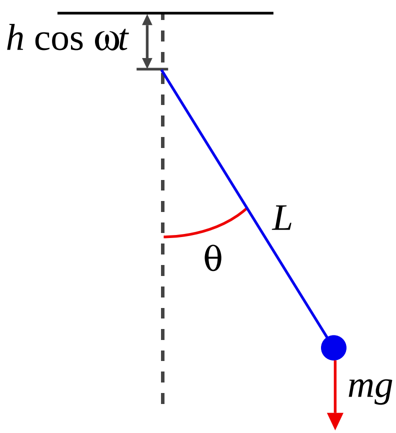
The driven plane pendulum

Definition:

• pendulum bob of mass m attached to rigid rod of length L and negligible mass;

• pendulum confined to swing in a plane;

• point of attachment of pendulum oscillates vertically with amplitude h and frequency ω.



Prerequisite:

• the simple pendulum with no driving force.

Why study it?

• it is one of the simplest dynamical systems exhibiting chaos.

Summary:

The equation of motion is

$$\boxed{\theta'' + \left(1 + H\Omega^2 \cos \Omega\tau\right)\sin\theta = 0} \quad ,$$

where

• $H = \dfrac{h}{L}$ • $\Omega = \dfrac{\omega}{\omega_0}$ • $\tau = \omega_0 t$,

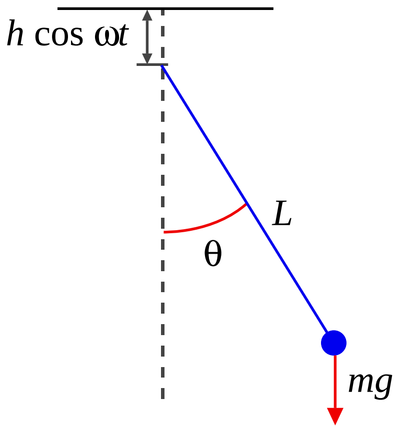
with the frequency of small oscillations of the unforced pendulum being $\omega_0 = \sqrt{\dfrac{g}{L}}$.

Go to derivation.

Go to Java™ applet

Chaotic motion

Let's modify an ordinary simple pendulum in an apparently innocent way. We'll attach the point of suspension of the pendulum to a motor, and make it go up and down with amplitude h and frequency ω :



With a fair amount of effort, you can show that the differential equation for θ has an extra term:

$$\ddot{\theta} + \omega_0^2 \left(1 + \frac{h\omega^2}{g} \cos \omega t \right) \sin \theta = 0 \ .$$

The point is that something odd happens. For some values of the initial conditions and h and ω, the motion is nice and regular. It may not be exactly periodic, but the pattern of its behavior is very predictable. Here is a movie showing a regular motion.

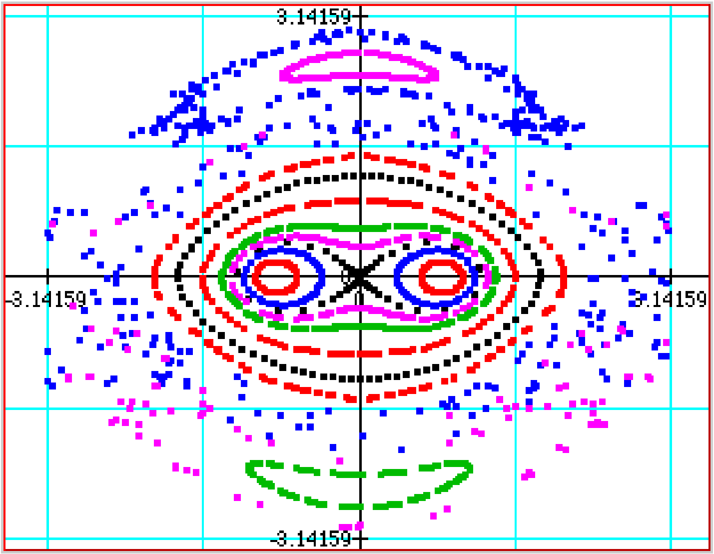
However, for other values of the initial conditions, the behavior is quite different. You wouldn't call it regular - *chaotic* is a more appropriate name. Here's a movie showing a chaotic motion.

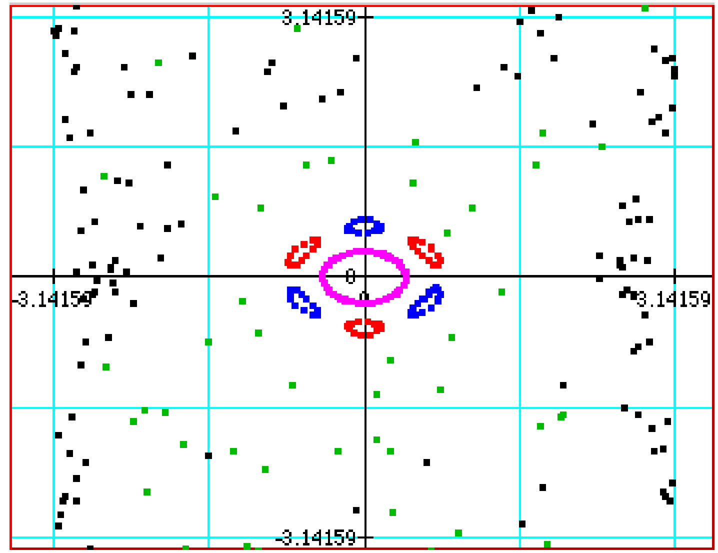
There's actually a precise way to define chaos, but we won't get into that here. Once chaos is defined, it is never easy to prove that any given motion is chaotic. It is also interesting (and hard!) to try to *predict* whether a given system will exhibit chaos.

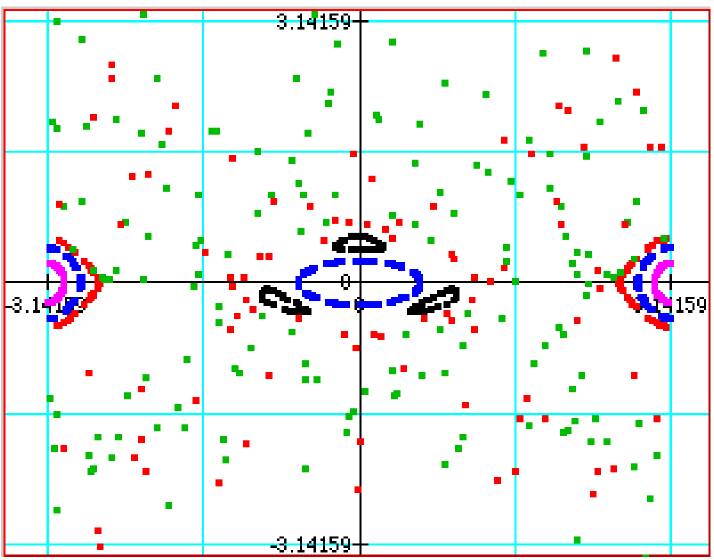
A very good way to see chaos is to form what is called a *Poincaré section*. Every time the pendulum's attachment point reaches the bottom, its angular velocity and angle are measured and a point is plotted in the *phase plane* (angular velocity versus angle).

For some values of the initial conditions and parameters, the resulting figure is very regular-looking. But for other values, the points are splashed around in the phase plane in a manner that is best described by the word "chaotic" - it is apparently quite random. Don't be fooled, however - such motion is quite predictable, since it follows from an ordinary differential equation.

Here are some pictures of Poincaré sections for the driven pendulum. It is fun to try to reproduce them using the Java™ applet.







The second of these three plots shows some trajectories which correspond to regular motions where the pendulum is pointing mostly upwards! And they all show some islands of regular motion in a sea of chaos.

This is a fascinating area of study, and belongs to a wider field called *nonlinear dynamics*. You may be interested to follow this link to the Los Alamos bulletin board, where the latest papers in this very active field are kept.

Here is a derivation of the equation of motion of the driven plane pendulum. It uses the Lagrangian formulation of mechanics.

The rectangular coordinates of the pendulum bob are

$$x = L\sin\theta \quad \text{and} \quad y = -(h(t) + L\cos\theta),$$

where *h(t)* is a function of time and *L* is constant.

The kinetic and potential energies are

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) \ \text{ and } \ V = mgy \ ,$$

where *m* is the mass of the bob and *g* is the gravitational acceleration. The Lagrangian is

$$\mathcal{L} = T - V,$$

and substituting the abovegives

$$\mathcal{L} = \frac{m}{2}(L^2\dot{\theta}^2 + \dot{h}^2 - 2L\dot{h}\dot{\theta}\sin\theta) + mg(h + L\cos\theta) \ .$$

The Euler-Lagrange equation of motion for the coordinate θ is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} \ .$$

Substituting, we find

$$\frac{d}{dt}\left(\frac{m}{2}(2L^2\dot{\theta} - 2L\dot{h}\sin\theta)\right) = -mL\dot{h}\dot{\theta}\cos\theta - mgL\sin\theta$$

After differentiating we find

$$\ddot{\theta} + \frac{g}{L}\left(1 - \frac{\ddot{h}}{g}\right)\sin\theta = 0 \ .$$

We recognize the square of the frequency of small oscillations of the unforced pendulum

$$\omega_0 = \sqrt{\frac{g}{L}} \ .$$

We let

$$h(t) = h \cos \omega t \ ,$$

where *h* is constant. Substituting in, we get

$$\ddot{\theta} + \omega_0^2\left(1 + \frac{h\omega^2}{g}\cos \omega t\right)\sin\theta = 0 \ .$$

It is convenient to convert to dimensionless variables

$$\tau = \omega_0 t\ , \ H = \frac{h}{L} \ , \ \text{ and } \ \Omega = \frac{\omega}{\omega_0} \ .$$

In terms of these, the equation of motion becomes

$$\theta'' + \left(1 + H\Omega^2\cos \Omega\tau\right)\sin\theta = 0 \ ,$$

where the primes denote differentiation with respect to τ.